

Landau-Lifshitz Equation with Affine Control

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Abstract

The Landau-Lifshitz equation is a coupled set of nonlinear partial differential equations that describes the dynamics of magnetization in a ferromagnet. This equation has an infinite number of stable equilibria. Steering the system from one equilibrium to another is a problem of both theoretical and practical interest. Since the objective is to steer between equilibria, approaches based on linearization are not appropriate. It is proven that affine proportional control can be used to steer the system from an arbitrary initial state, including an equilibrium point, to a specified equilibrium point. The second point becomes a globally asymptotically stable equilibrium of the controlled system. The control also removes hysteresis from the Landau-Lifshitz equation. These results are illustrated with simulations.

Keywords: Asymptotic stability; Exponential stability; Hysteresis Loops; Lyapunov function; Linear control systems; Partial differential equations

1. Introduction

The Landau-Lifshitz equation was developed to model the behaviour of domain walls in magnetic regions within ferromagnetic structures [1]. For example, the one-dimensional Landau-Lifshitz equation can be used to describe ferromagnetic nanowires, which are often found in memory storage devices such as hard disks, credit cards or tape recordings. Each set of data stored in a memory device is uniquely assigned to a specific stable magnetic state of the ferromagnet. This can be difficult to achieve due to the presence of hysteresis. Hysteresis is characterized by the presence of multiple equilibria, and looping in the input-output map is typical [2, 3]. Consequently, a particular input can lead to different magnetizations. Therefore, it is desirable to control magnetization between different stable equilibria.

The Landau-Lifshitz equation is known to exhibit hysteretic behaviour. For example, [4, 5] investigated via experiments the shape change of the hysteresis loop as the structure of the nanomagnet is varied. Experiments conducted on nanowires also demonstrate hysteresis loops [6]. Numerical simulations illustrating hysteresis loops are found in [7, 8]. The dynamics of hysteresis in the Landau-Lifshitz equation has also been represented by a hysteresis operator [9, 10]. In much of the aforementioned literature, the presence of hysteresis in the Landau-Lifshitz equation is identified by the fact that input-output curves exhibit a looping behaviour. This alone is not enough to characterize hysteresis [2, 3, 11]. A looping behaviour must persist with low frequency periodic inputs.

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Definition 1. [11] A system exhibits *hysteresis* if a nontrivial closed curve in the input–output map persists for a periodic input as the frequency component of the input signal approaches zero.

Another approach to hysteresis is based on the existence of multiple stable equilibria, which are present in the (uncontrolled) Landau-Lifshitz equation [12, Chapter 6].

Definition 2. [3, Definition 3] A system exhibits *hysteresis* if it has

- (a) multiple stable equilibrium points and
- (b) dynamics that are considerably faster than the time scale at which inputs are varied.

Note that condition (b) is relative to the speed at which a controlled input is changed. In many cases, hysteresis is present but is rate-dependent [3].

There is now an extensive body of results on control and stabilization of linear partial differential equations (PDEs); see for instance the books [13, 14, 15, 16] and the review paper [17]. There are fewer results on control and stabilization of nonlinear partial differential equations and the Landau-Lifshitz equation is particularly problematic. Stability of the Landau-Lifshitz equation is often based on linearization [18, 19, 20, 21, 22]. Local asymptotic stability is shown in [23] for the controlled linearized Landau-Lifshitz equation. However, because the Landau-Lifshitz equation is not quasi-linear, analysis based on a linearization may not predict stability of the original system; see [24, 25, 26]. Also, when the goal is to steer between equilibria, global stability results are needed. Experiments and numerical simulations on the control of domain walls in a nanowire are presented in [6, 27]. In [28, 29], solutions to the Landau-Lifshitz equation are shown to be arbitrarily close to domain walls given a constant control.

In the next section, the uncontrolled Landau-Lifshitz equation and its equilibrium points are described. In [2], simulations were used to show the Landau-Lifshitz and the linearized Landau-Lifshitz equation exhibit hysteresis. This suggests hysteresis is not due entirely to nonlinearity. These results are reviewed in Section 2. Theorem 4 demonstrates the linearized uncontrolled Landau-Lifshitz equation has a zero eigenvalue. This suggests use of a proportional controller to stabilize the equation about a given point. It is then proven in Section 3 that stabilization of the full Landau-Lifshitz equation is achieved with a proportional affine control. Proportional control can be used to steer the system to an arbitrary equilibrium point of the uncontrolled equation; in fact, the system can be steered between these points. Simulations illustrating these results are presented in Section 4. Furthermore, simulations indicate that hysteresis is absent in the controlled system.

2. Landau-Lifshitz Equation and Hysteresis

Consider the magnetization

$$\mathbf{m}(x, t) = (m_1(x, t), m_2(x, t), m_3(x, t)),$$

at position $x \in [0, L]$ and time $t \geq 0$ in a long thin ferromagnetic material of length $L > 0$. If only the exchange energy term is considered, the magnetization is modelled by the one-dimensional (uncontrolled) Landau-Lifshitz equation [30],[12, Chapter 6]

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) \quad (1a)$$

$$\mathbf{m}(x, 0) = \mathbf{m}_0(x) \quad (1b)$$

$$\mathbf{m}_x(0, t) = \mathbf{m}_x(L, t) = \mathbf{0}. \quad (1c)$$

where \times denotes the cross product and $\nu \geq 0$ is the damping parameter, which depends on the type of ferromagnet. The notation \mathbf{m}_x and \mathbf{m}_{xx} means the magnetization is differentiated with respect to x once and twice, respectively. The gyromagnetic ratio multiplying $\mathbf{m} \times \mathbf{m}_{xx}$ has been normalized to 1 for simplicity. For details on the damping parameter and gyromagnetic ratio, see [31]. It is assumed there is no magnetic flux at the boundaries and so Neumann boundary conditions are appropriate.

Define $\mathcal{L}_2^3 = \mathcal{L}_2([0, L]; \mathbb{R}^3)$ with the usual inner product and norm, denoted $\|\cdot\|_{\mathcal{L}_2^3}$, and the operator

$$f(\mathbf{m}) = \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}), \quad (2)$$

and its domain

$$D = \{\mathbf{m} \in \mathcal{L}_2^3 : \mathbf{m}_x \in \mathcal{L}_2^3, \mathbf{m}_{xx} \in \mathcal{L}_2^3, \mathbf{m}_x(0) = \mathbf{m}_x(L) = \mathbf{0}\}. \quad (3)$$

The following theorem is a consequence of the existence and uniqueness results in [32, 33].

Theorem 1. *If $m(x, 0) \in \mathcal{L}_2^3$, then the operator $f(\mathbf{m})$ with domain D generates a nonlinear contraction semigroup on \mathcal{L}_2^3 .*

Ferromagnets are magnetized to saturation [34, Section 4.1]; that is

$$\|\mathbf{m}_0(x)\|_2 = M_s$$

where $\|\cdot\|_2$ is the Euclidean norm and M_s is the magnetization saturation. Physically, this means that at each point, x , the magnitude of $\mathbf{m}_0(x)$ equals the magnetization saturation. In much of the literature, M_s is set to 1; see for example, [12, Section 6.3.1], [32, 33, 35]. This convention is used here. The magnitude of the magnetization does not change with time.

Theorem 2. [12, Lemma 6.3.1] *If $\|\mathbf{m}_0(x)\|_2 = 1$, then for all $t \geq 0$ the solution to (1a) satisfies*

$$\|\mathbf{m}(x, t)\|_2 = 1. \quad (4)$$

The initial condition $\mathbf{m}_0(x)$ is assumed to be real-valued, which implies $m(x, t)$ is real-valued for all time.

The set of equilibrium points of (1) is [12, Theorem 6.1.1]

$$E = \{\mathbf{a} = (a_1, a_2, a_3) : a_1, a_2, a_3 \text{ constants and } \mathbf{a}^T \mathbf{a} = 1\}. \quad (5)$$

In [12, Proposition 6.2.1], the stability of the equilibrium points is established using Lyapunov's Theorem and the Lyapunov function

$$V(\mathbf{m}) = \frac{1}{2} \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2.$$

Furthermore, E is an asymptotically stable equilibrium set, as stated in the following theorem. The proof is the same as that in [12, Proposition 6.2.1] except it is for equilibrium sets, rather than equilibrium points. However, individual equilibrium points are only stable, not asymptotically stable. Control is needed to obtain asymptotic stability as illustrated in Section 3.

Theorem 3. *The equilibrium set in (5) is asymptotically stable in the \mathcal{L}_2^3 -norm.*

The existence of multiple stable equilibria indicates the presence of hysteresis in the Landau-Lifshitz equation (care of Definition 2). Definition 1 is used to establish hysteresis in simulations of the Landau-Lifshitz equation. For the simulations, a Galerkin approximation for the Landau-Lifshitz equation using linear spline elements is used. The number of elements is 5 and a periodic input, $\hat{\mathbf{u}} = (0, 0.001 \cos(\omega t), 0)$, is applied to the Landau-Lifshitz equation to construct the input-output map. Plots of $\mathbf{m}(x, t)$ with x fixed against the periodic input are illustrated in Figure 1 for varying frequencies ω . It is clear from Figure 1 the input-output curves exhibit persistent looping behaviour as the frequency of the input approaches zero. The continuum of equilibrium points explains the absence of sharp jumps that often appear in hysteresis loops. The similar appearance of the loop shapes between $m_1(x, t), m_2(x, t), m_3(x, t)$ is due to the symmetric structure of the Landau-Lifshitz equation.

To obtain the linear uncontrolled Landau-Lifshitz equation, equation (1a) is first rewritten in semilinear form,

$$\frac{\partial \mathbf{m}}{\partial t} = \nu \mathbf{m}_{xx} + \mathbf{m} \times \mathbf{m}_{xx} + \nu \|\mathbf{m}_x\|_2^2 \mathbf{m}, \quad (6)$$

using equation (4) and properties of cross products, and then $\mathbf{m}(x, t) = \mathbf{a} + \mathbf{z}(x, t)$ is substituted into (6) where $\mathbf{a} \in E$ is an equilibrium of (1) and $\mathbf{z} \in \mathcal{L}_2^3$ is a small perturbation. The Landau-Lifshitz equation linearized about an equilibrium \mathbf{a} is

$$\dot{\mathbf{z}} = A\mathbf{z}, \quad \mathbf{z}(0) = \mathbf{z}_0 \quad (7a)$$

$$\mathbf{z}_x(0, t) = \mathbf{0} = \mathbf{z}_x(L, t) \quad (7b)$$

where A is the linear operator,

$$A\mathbf{z} = \nu \mathbf{z}_{xx} + \mathbf{a} \times \mathbf{z}_{xx},$$

and the domain is

$$D(A) = \{\mathbf{z} : \mathbf{z} \in \mathcal{L}_2^3, \mathbf{z}_x \in \mathcal{L}_2^3, \mathbf{z}_{xx} \in \mathcal{L}_2^3, \mathbf{z}_x(0, t) = \mathbf{0} = \mathbf{z}_x(L, t)\},$$

Using [36, Theorem 6.2], the linear operator A can be shown to generate an analytic semigroup; for details, see [37, Theorem 4.16].

Theorem 4. [2, Theorem 5] Any constant $\mathbf{c} \in \mathbb{R}^3$ is a stable equilibrium of (7).

PROOF. For completeness, the proof is included here. Since A generates an analytic semigroup, the spectrum determined growth assumption is satisfied and so the eigenvalues of A determine the stability of the linear system (7) [14, Section 5.1], [38, Section 3.2].

It is clear that any constant function \mathbf{c} is an equilibrium of (7). Let $\lambda \in \mathbb{C}$. The eigenvalue problem of (7) is $\lambda \mathbf{v} = A\mathbf{v}$ and boundary conditions $\mathbf{v}_x(0) = \mathbf{v}_x(L) = \mathbf{0}$ where $\mathbf{v} \in \mathcal{L}_2^3$. Solving, the eigenvalues of (7) are the zero eigenvalue, $\lambda_1 = 0$, which is associated to a nonzero constant eigenvector, and the remaining eigenvalues are of the form

$$\begin{aligned} \lambda_2^{+,-} &= \frac{-(1+2n)^2 \pi^2 \nu}{L^2} \pm i \frac{(1+2n)^2 \pi^2}{L^2}, & \lambda_3 &= \frac{-(1+2n)^2 \pi^2 \nu}{L^2}, \\ \lambda_4^{+,-} &= \frac{-(2n)^2 \pi^2 \nu}{L^2} \pm i \frac{(2n)^2 \pi^2}{L^2}, & \lambda_5 &= \frac{-(2n)^2 \pi^2 \nu}{L^2} \end{aligned}$$

where $n \in \mathbb{Z}$. Since all the eigenvalues have nonpositive real part, the equilibria of (7) are stable. \square

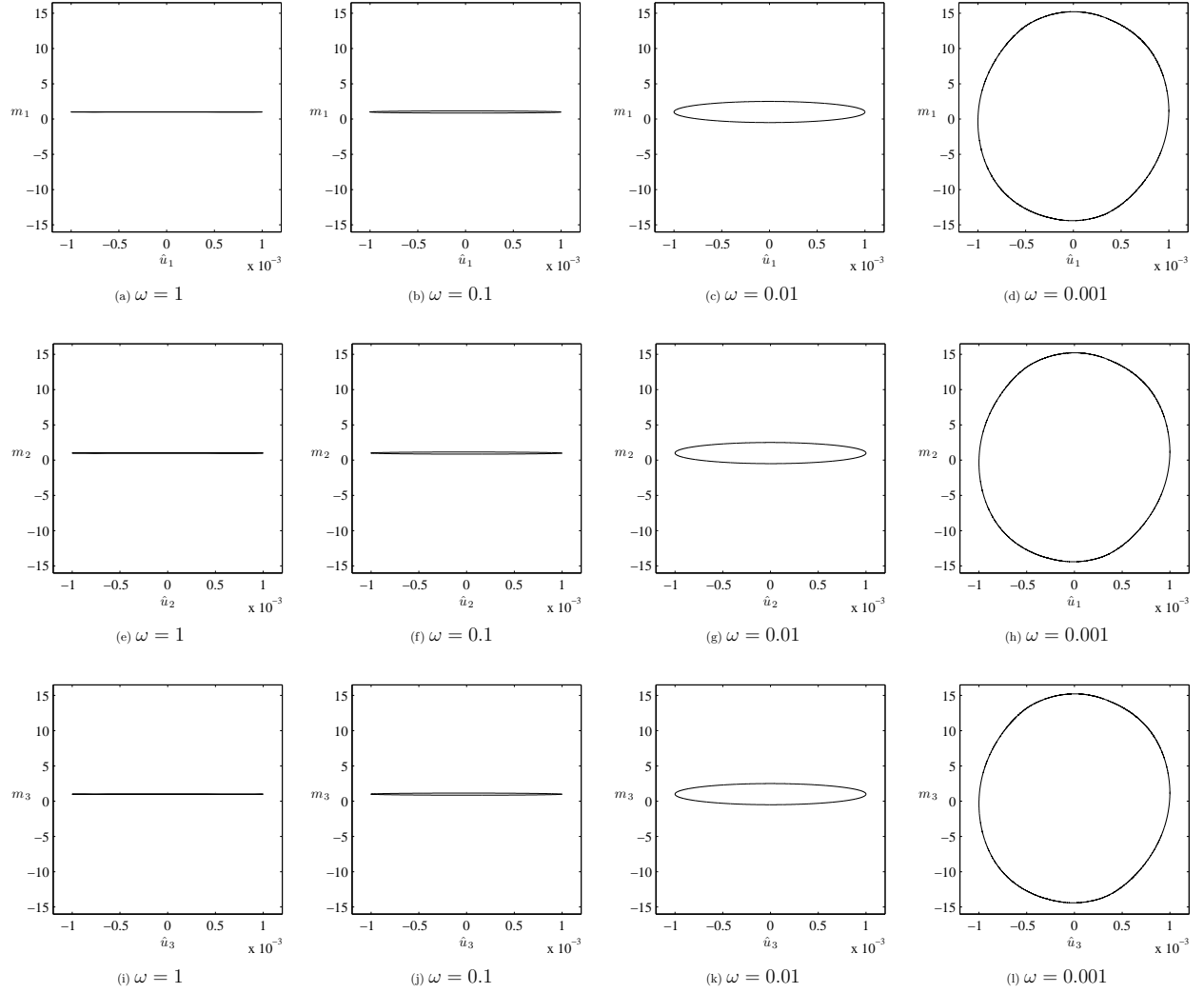


Figure 1: Input-output curves for the (nonlinear) Landau-Lifshitz equation demonstrate persistent looping behaviour as the frequency of the periodic input, $\hat{\mathbf{u}}$, approaches zero and hence indicates the presence of hysteresis. (a)–(d) Input-output curves for $m_1(x, t)$ with $\hat{\mathbf{u}} = (0.001 \cos(\omega t), 0, 0)$ and $\mathbf{m}_0(x) = (1, 0, 0)$. (e)–(h) Input-output curves for $m_2(x, t)$ with $\hat{\mathbf{u}} = (0, 0.001 \cos(\omega t), 0)$ and $\mathbf{m}_0(x) = (0, 1, 0)$. (i)–(l) Input-output curves for $m_3(x, t)$ with $\hat{\mathbf{u}} = (0, 0, 0.001 \cos(\omega t))$ and $\mathbf{m}_0(x) = (0, 0, 1)$. ($L = 1$, $\nu = 0.02$, $x = 0.6$)

Using Theorem 4 and Definition 2 indicates the linear Landau-Lifshitz equation exhibits hysteresis. Furthermore, simulations with periodic inputs were performed to determine whether persistent loops exist and hence Definition 1 is satisfied. Again, $\nu = 0.02$, $L = 1$, and the same periodic input is applied as for the nonlinear Landau-Lifshitz equation. Figure 2 shows the input-output curves for the first component of the solution to the linear Landau-Lifshitz equation with $\mathbf{a} = (1, 0, 0)$ and initial condition $\mathbf{z}_0(x) = (1, 0, 0)$. From the figure, it is clear a loop persists as the frequency of the input approaches zero. Similar plots are obtained when the control is on the second and third components. The hysteresis loops in Figure 2 are similar in shape to the nonlinear Landau-Lifshitz equation depicted in Figure 1, both of which have a continuum of equilibria.

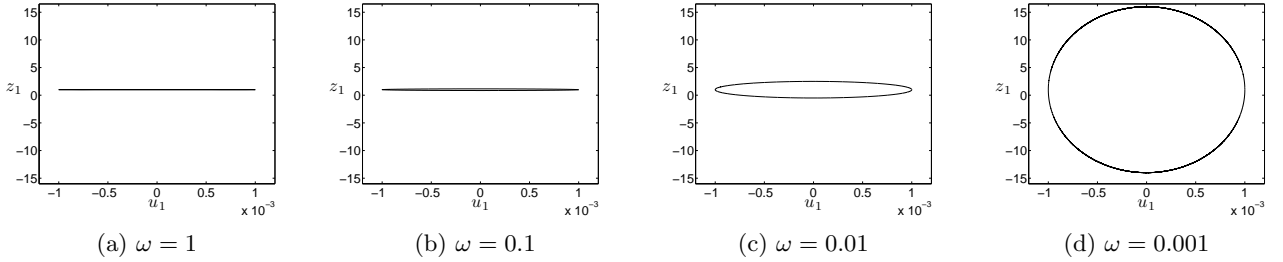


Figure 2: Hysteresis loops for $z_1(x, t)$ of the linear Landau-Lifshitz equation with $x = 0.6$ and $\nu = 0.02$. The linearization is at $\mathbf{a} = (1, 0, 0)$. The input is $\mathbf{u}(t) = (0.001 \cos(\omega t), 0, 0)$ and the initial condition is $\mathbf{z}_0(x) = (1, 0, 0)$.

3. Controller Design

A control, $\mathbf{u}(t)$, is introduced into the Landau-Lifshitz equation (1a) as follows

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial t} &= \mathbf{m} \times \mathbf{m}_{xx} - \nu \mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx}) + \mathbf{u}(t) \\ \mathbf{m}(x, 0) &= \mathbf{m}_0(x) \\ \mathbf{m}_x(0, t) &= \mathbf{m}_x(L, t) = \mathbf{0}. \end{aligned} \tag{8}$$

The goal is to choose a control $\mathbf{u}(t)$ so the system governed by the Landau-Lifshitz equation moves from an arbitrary initial condition, possibly an equilibrium point, to a specified equilibrium point \mathbf{r} , where $\mathbf{r} \in E$ and E is defined in (5). The control function needs to be chosen so that \mathbf{r} becomes an asymptotically stable equilibrium point of the controlled system.

Theorem 4 implies zero is an eigenvalue of the uncontrolled linearized Landau-Lifshitz equation. For finite-dimensional linear systems, simple proportional control of a system with a zero eigenvalue yields asymptotic tracking of a specified state and this motivates choosing the control

$$\mathbf{u}(x, t) = k(\mathbf{r} - \mathbf{m}(x, t)) \tag{9}$$

where k is a positive constant control parameter for equation (8). It is clear that \mathbf{r} is an equilibrium point of (8) with the control in (9). Figure 3 is a block diagram representation of (8) with control (9).

The following theorem establishes well-posedness of the controlled equation. In particular, for any initial condition \mathbf{m}_0 , the solution to (8) with control $u(t) = k(\mathbf{r} - \mathbf{m})$ satisfies $\|\mathbf{m}(\cdot, t)\|_{\mathcal{L}_2^3} \leq 1$.

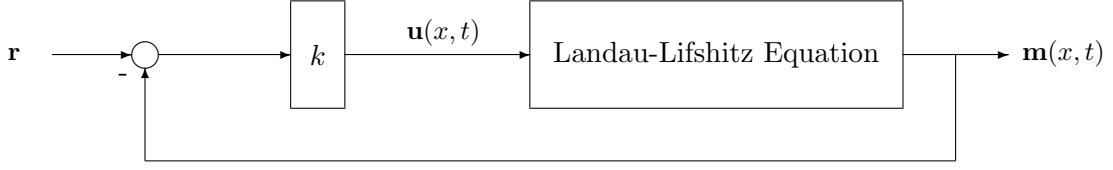


Figure 3: Closed-loop control of the Landau-Lifshitz equation.

Theorem 5. For any $\mathbf{r} \in E$, define the operator

$$B\mathbf{m} = k(\mathbf{r} - \mathbf{m}). \quad (10)$$

If $k > 0$, the nonlinear operator $f + B$ with domain D , where f and D are defined in (2), (3) respectively, generates a nonlinear contraction semigroup on \mathcal{L}_2^3 .

PROOF. (i) For any $\mathbf{m}, \mathbf{y} \in D$,

$$\begin{aligned} & \langle f(\mathbf{m}) + B\mathbf{m} - (f(\mathbf{y}) + B\mathbf{y}), \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} \\ &= \langle f(\mathbf{m}) - f(\mathbf{y}), \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} + \langle B\mathbf{m} - B\mathbf{y}, \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3}. \end{aligned}$$

Since f generates a nonlinear contraction semigroup (Theorem 1), then f is dissipative [38, Proposition 2.98]; that is,

$$\langle f(\mathbf{m}) - f(\mathbf{y}), \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} \leq 0.$$

It follows that

$$\begin{aligned} \langle f(\mathbf{m}) + B\mathbf{m} - (f(\mathbf{y}) + B\mathbf{y}), \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} &\leq \langle B\mathbf{m} - B\mathbf{y}, \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} \\ &= \langle -k\mathbf{m} + k\mathbf{y}, \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} \\ &= -k \langle \mathbf{m} - \mathbf{y}, \mathbf{m} - \mathbf{y} \rangle_{\mathcal{L}_2^3} \\ &\leq 0 \end{aligned}$$

and hence $f + B$ is dissipative.

(ii) Since f generates a nonlinear contraction semigroup (Theorem 1), $\text{ran}(\mathbf{I} - \hat{\alpha}f) = \mathcal{L}_2^3$ for any $\hat{\alpha} > 0$ [39, Lemma 2.1]. This means that for any $\mathbf{y}_2 \in \mathcal{L}_2^3$ there exists $\mathbf{m} \in D$ such that $\mathbf{m} - \hat{\alpha}f(\mathbf{m}) = \mathbf{y}_2$. Choose any $\mathbf{y}_1 \in \mathcal{L}_2^3$, $\alpha > 0$ and define

$$\mathbf{y}_2 = \frac{\mathbf{y}_1}{1 + \alpha k} + \frac{\alpha k \mathbf{r}}{1 + \alpha k}$$

and

$$\hat{\alpha} = \frac{\alpha}{1 + \alpha k}.$$

There exists $\mathbf{m} \in D$ such that

$$\mathbf{m} - \frac{\alpha}{1 + \alpha k} f(\mathbf{m}) = \mathbf{y}_2 = \frac{\mathbf{y}_1}{1 + \alpha k} + \frac{\alpha k \mathbf{r}}{1 + \alpha k}.$$

Solving for \mathbf{y}_1 leads to

$$\mathbf{y}_1 = \mathbf{m} - \alpha(k(\mathbf{r} - \mathbf{m}) + f(\mathbf{m})).$$

Thus, for any $\mathbf{y}_1 \in \mathcal{L}_2^3$, there exists $\mathbf{m} \in D$ such that $\mathbf{y}_1 = (\mathbf{I} - \alpha(B + f))\mathbf{m}$ and hence $\text{ran}(\mathbf{I} - \alpha(B + f)) = \mathcal{L}_2^3$ for some $\alpha > 0$. It follows that the range is \mathcal{L}_2^3 for all $\alpha > 0$ [39, Lemma 2.1].

Thus, since $B + f$ is dissipative and the range of $(\mathbf{I} - \alpha(B + f))$ is \mathcal{L}_2^3 , then $B + f$ generates a nonlinear contraction semigroup [38, Proposition 2.114]. \square

The following lemmas are needed in the proof of the main results in Theorem 11 and Theorem 12. The first theorem demonstrates the control in (9) can steer the dynamics in the Landau-Lifshitz to an asymptotically stable state in the L_2^3 -norm, while the latter theorem establishes exponential stability in the H_1 -norm, $\|\mathbf{m}\|_{H_1}^2 = \|\mathbf{m}\|_{\mathcal{L}_2^3}^2 + \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2$.

Lemma 6. *If $\mathbf{a} \in E$ where E is defined in (5), then $\|\mathbf{a} \times \mathbf{m}\|_{\mathcal{L}_2^3} \leq \|\mathbf{m}\|_{\mathcal{L}_2^3}$ for all $\mathbf{m} \in \mathcal{L}_2^3$.*

PROOF. Since $\|\mathbf{a} \times \mathbf{m}\|_2 = \|\mathbf{a}\|_2 \|\mathbf{m}\|_2 \sin(\theta)$ where θ is the angle between \mathbf{a} and \mathbf{m} , and $\|\mathbf{a}\|_2 = 1$, then $\|\mathbf{a} \times \mathbf{m}\|_2 \leq \|\mathbf{m}\|_2$. Extending to the L_2^3 -norm, the desired result is obtained.

Lemmas 7 and 8 are simple consequences of the product rule.

Lemma 7. *For $\mathbf{m} \in \mathcal{L}_2^3$, the derivative of $\mathbf{g} = \mathbf{m} \times \mathbf{m}_x$ is $\mathbf{g}_x = \mathbf{m} \times \mathbf{m}_{xx}$.*

Lemma 8. *For $\mathbf{m} \in \mathcal{L}_2^3$, the derivative of $f = (\mathbf{m} \times \mathbf{m}_x)^T (\mathbf{m} \times \mathbf{m}_x)$ is $f_x = 2(\mathbf{m} \times \mathbf{m}_x)^T (\mathbf{m} \times \mathbf{m}_{xx})$.*

Lemma 9. *For $\mathbf{m} \in \mathcal{L}_2^3$ satisfying (1c),*

$$\int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx = 0.$$

PROOF. Integrating by parts, and applying Lemma 7 and the boundary conditions (1c) implies

$$\int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx = - \int_0^L \mathbf{m}_x^T (\mathbf{m} \times \mathbf{m}_x) dx.$$

From properties of cross products, $\mathbf{m}_x^T (\mathbf{m} \times \mathbf{m}_x) = \mathbf{m}^T (\mathbf{m}_x \times \mathbf{m}_x) = 0$, and hence the integral is zero. \square

Lemma 10. *For $\mathbf{m} \in \mathcal{L}_2^3$ satisfying (1c),*

$$\|\mathbf{m} \times \mathbf{m}_x\|_{\mathcal{L}_2^3} \leq 4L^2 \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}$$

PROOF. Integrating by parts, using Lemma 8 and the boundary conditions (1c) leads to

$$\|\mathbf{m} \times \mathbf{m}_x\|_{\mathcal{L}_2^3}^2 = - \int_0^L 2(\mathbf{m} \times \mathbf{m}_x)^T (\mathbf{m} \times \mathbf{m}_{xx}) x dx.$$

It follows from Young's inequality that

$$\|\mathbf{m} \times \mathbf{m}_x\|_{\mathcal{L}_2^3}^2 \leq \frac{1}{2} \int_0^L (\mathbf{m} \times \mathbf{m}_x)^T (\mathbf{m} \times \mathbf{m}_x) dx + \int_0^L 2(\mathbf{m} \times \mathbf{m}_{xx})^T (\mathbf{m} \times \mathbf{m}_{xx}) x^2 dx.$$

Since $x \in [0, L]$,

$$\|\mathbf{m} \times \mathbf{m}_x\|_{\mathcal{L}_2^3}^2 \leq \frac{1}{2} \|\mathbf{m} \times \mathbf{m}_x\|_{\mathcal{L}_2^3}^2 + 2L^2 \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2.$$

Rearranging gives the desired inequality. \square

Theorem 11. *Let \mathbf{r} be an equilibrium point of (8) with control defined in (9). For any positive constant k such that $k > 8\nu L^4$, \mathbf{r} is a globally asymptotically stable point of (8) in the \mathcal{L}_2^3 -norm.*

PROOF. The Lyapunov candidate is

$$V(\mathbf{m}) = \frac{1}{2} \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 + \frac{1}{2} \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2$$

which is clearly nonnegative. Furthermore, $V = 0$ if and only if $\mathbf{m} = \mathbf{r}$. Taking the derivative of V

$$\begin{aligned} \frac{dV}{dt} &= \int_0^L (\mathbf{m} - \mathbf{r})^T \dot{\mathbf{m}} dx + \int_0^L \mathbf{m}_x^T \dot{\mathbf{m}}_x dx \\ &= \int_0^L (\mathbf{m} - \mathbf{r})^T \dot{\mathbf{m}} dx - \int_0^L \mathbf{m}_{xx}^T \dot{\mathbf{m}} dx \end{aligned}$$

where the dot notation means differentiation with respect to t . Substituting in (8) to eliminate $\dot{\mathbf{m}}$,

$$\begin{aligned} \frac{dV}{dt} &= \int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx - \nu \int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})) dx \\ &\quad + k \int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{r} - \mathbf{m}) dx - \int_0^L \mathbf{m}_{xx}^T (\mathbf{m} \times \mathbf{m}_{xx}) dx \\ &\quad + \nu \int_0^L \mathbf{m}_{xx}^T (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})) dx - k \int_0^L \mathbf{m}_{xx}^T (\mathbf{r} - \mathbf{m}) dx. \end{aligned}$$

From Lemma 9, the first integral is zero. Furthermore, from properties of cross products,

$$\mathbf{m}_{xx}^T (\mathbf{m} \times \mathbf{m}_{xx}) = \mathbf{m}^T (\mathbf{m}_{xx} \times \mathbf{m}_{xx}) = 0,$$

and hence

$$\int_0^L \mathbf{m}_{xx}^T (\mathbf{m} \times \mathbf{m}_{xx}) dx = 0.$$

It follows that

$$\begin{aligned} \frac{dV}{dt} &= -\nu \int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})) dx - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 \\ &\quad - \nu \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2 - k \int_0^L \mathbf{m}_{xx}^T (\mathbf{r} - \mathbf{m}) dx. \end{aligned}$$

Applying integration by parts to the last integral leads to

$$\begin{aligned} \frac{dV}{dt} &= -\nu \int_0^L (\mathbf{m} - \mathbf{r})^T (\mathbf{m} \times (\mathbf{m} \times \mathbf{m}_{xx})) dx - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 - \nu \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 \\ &= -\nu \int_0^L ((\mathbf{m} - \mathbf{r}) \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 - \nu \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 \\ &= \nu \int_0^L (\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 - \nu \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2. \end{aligned} \tag{11}$$

Applying integration by parts with Lemma 7 to the integral implies

$$\int_0^L (\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx = \left[(\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_x) \right]_0^L - \int_0^L (\mathbf{r} \times \mathbf{m}_x)^T (\mathbf{m} \times \mathbf{m}_x) dx$$

and substituting in the boundary conditions in (1c) leads to

$$\int_0^L (\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx = -\langle \mathbf{r} \times \mathbf{m}_x, \mathbf{m} \times \mathbf{m}_x \rangle_{\mathcal{L}_2^3}.$$

Then from Cauchy-Schwarz and Lemma 10,

$$\begin{aligned} \int_0^L (\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx &\leq \|\mathbf{r} \times \mathbf{m}_x\|_{\mathcal{L}_2^3} \|\mathbf{m} \times \mathbf{m}_x\|_{\mathcal{L}_2^3} \\ &\leq 4L^2 \|\mathbf{r} \times \mathbf{m}_x\|_{\mathcal{L}_2^3} \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}. \end{aligned}$$

It follows from Young's Inequality that

$$\int_0^L (\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx \leq 8L^4 \|\mathbf{r} \times \mathbf{m}_x\|_{\mathcal{L}_2^3}^2 + \frac{1}{2} \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2$$

and from Lemma 6,

$$\int_0^L (\mathbf{r} \times \mathbf{m})^T (\mathbf{m} \times \mathbf{m}_{xx}) dx \leq 8L^4 \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 + \frac{1}{2} \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2$$

Substituting this result into (11) leads to

$$\frac{dV}{dt} \leq -(k - 8\nu L^4) \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 - \frac{\nu}{2} \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2. \quad (12)$$

The derivative is negative if $k > 8\nu L^4$. It follows that

$$\frac{dV}{dt} \leq -k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2.$$

Therefore, $dV/dt < 0$ for all $\mathbf{m} \neq \mathbf{r}$ and $dV/dt = 0$ if $\mathbf{m} = \mathbf{r}$. Since $V(\mathbf{m}) \geq \frac{1}{2} \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2$, $V \rightarrow \infty$ as $\|\mathbf{m} - \mathbf{r}\| \rightarrow \infty$. From Lyapunov's Theorem [40, Theorem 6.2.13], \mathbf{r} is a globally asymptotically stable equilibrium of (8). \square

Theorem 12. *Let \mathbf{r} be an equilibrium point of (8) with control defined in (9). For any positive constant k such that $k > 8\nu L^4$, \mathbf{r} is a globally exponentially stable equilibrium point of (8) in the H_1 -norm. That is, for any initial condition on H_1 , \mathbf{m} decreases exponentially in the H_1 -norm to \mathbf{r} .*

PROOF. In the proof of Theorem 11, we have from (12) that the derivative of V satisfies

$$\frac{dV}{dt} \leq -(k - 8\nu L^4) \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 - \frac{\nu}{2} \|\mathbf{m} \times \mathbf{m}_{xx}\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2$$

and hence

$$\begin{aligned}\frac{dV}{dt} &\leq -(k - 8\nu L^4) \|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 \\ &\leq -(k - 8\nu L^4) \left(\|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 + \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 \right) \\ &= -2(k - 8\nu L^4) V.\end{aligned}$$

Integrating with respect to time

$$\|\mathbf{m}_x\|_{\mathcal{L}_2^3}^2 + \|\mathbf{m} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 \leq e^{-2(k-8\nu L^4)t} \left(\|\mathbf{m}_x(x, 0)\|_{\mathcal{L}_2^3}^2 + \|\mathbf{m}(x, 0) - \mathbf{r}\|_{\mathcal{L}_2^3}^2 \right)$$

Noting that \mathbf{r} does not depend on x , it follows that

$$\|\mathbf{m} - \mathbf{r}\|_{H_1}^2 \leq e^{-2(k-8\nu L^4)t} \|\mathbf{m}(x, 0) - \mathbf{r}\|_{H_1}^2$$

and since $k > 8\nu L^4$, \mathbf{r} is an exponentially stable equilibrium point of (8). \square

A natural question is whether \mathbf{r} is exponentially stable in the \mathcal{L}_2^3 -norm. Analysis of the linear Landau-Lifshitz equation provides insight to this question. For the control in (9), the linearized controlled Landau-Lifshitz equation is

$$\frac{\partial \mathbf{z}}{\partial t} = \nu \mathbf{z}_{xx} + \mathbf{a} \times \mathbf{z}_{xx} + k(\mathbf{r} - \mathbf{z}), \quad \mathbf{z}(0) = \mathbf{z}_0 \quad (13)$$

with the same boundary conditions $\mathbf{z}_x(0) = \mathbf{z}_x(L) = \mathbf{0}$. Since the uncontrolled linear Landau-Lifshitz equation (7) generates a linear semigroup and $k(\mathbf{r} - \mathbf{z})$ is a bounded linear (affine) operator, then the operator in (13) generates a semigroup [14, Theorem 3.2.1]. Substituting $\mathbf{z} = \mathbf{r}$ into (13) leads to $\partial \mathbf{z} / \partial t = \mathbf{0}$ and hence \mathbf{r} is a stable equilibrium point of (13).

Theorem 13. *Let $\mathbf{r} \in E$. For any positive constant k , \mathbf{r} is an exponentially stable equilibrium of the linearized system (13) in the \mathcal{L}_2^3 -norm.*

PROOF. For $\mathbf{z} \in D(A)$, where $D(A) = D$ as in equation (3), consider the Lyapunov candidate

$$V(\mathbf{z}) = \frac{1}{2} \|\mathbf{z} - \mathbf{r}\|_{\mathcal{L}_2^3}^2.$$

It is clear that $V \geq 0$ for all $\mathbf{z} \in D(A)$ and furthermore, $V(\mathbf{z}) = 0$ only when $\mathbf{z} = \mathbf{r}$. Therefore, $V(\mathbf{z}) > 0$ for all $\mathbf{z} \in D(A) \setminus \{\mathbf{r}\}$.

Taking the derivative of $V(\mathbf{z})$ implies

$$\frac{dV}{dt} = \int_0^L (\mathbf{z} - \mathbf{r})^T \dot{\mathbf{z}} dx.$$

Substituting in (13) yields

$$\frac{dV}{dt} = \nu \int_0^L (\mathbf{z} - \mathbf{r})^T \mathbf{z}_{xx} dx + \int_0^L (\mathbf{r} - \mathbf{z})^T (\mathbf{a} \times \mathbf{z}_{xx}) dx + k \int_0^L (\mathbf{z} - \mathbf{r})^T (\mathbf{r} - \mathbf{z}) dx.$$

By Lemma 9, the middle term is zero. Using integration by parts, the first term becomes

$$-\nu \int_0^L \mathbf{z}_x^T \mathbf{z}_x dx.$$

It follows that

$$\frac{dV}{dt} = -\nu \|\mathbf{z}_x\|_{\mathcal{L}_2^3}^2 - k \|\mathbf{z} - \mathbf{r}\|_{\mathcal{L}_2^3}^2$$

and since $\nu \geq 0$,

$$\frac{dV}{dt} \leq -k \|\mathbf{z} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 = -2kV.$$

Solving yields

$$\|\mathbf{z} - \mathbf{r}\|_{\mathcal{L}_2^3}^2 \leq e^{-2kt} \|\mathbf{z}_0 - \mathbf{r}\|_{\mathcal{L}_2^3}^2.$$

For $k > 0$ the equilibrium point, \mathbf{r} , of (13) is exponentially stable. \square

Theorem 13 suggests the equilibrium point in the controlled nonlinear Landau-Lifshitz equation (8) is exponentially stable in the \mathcal{L}_2^3 -norm. However, since the nonlinearity in the Landau-Lifshitz equation is unbounded, stability of the linear equation does not necessarily reflect stability of the original nonlinear equation; see [24, 25, 26].

4. Example

Simulations illustrating the stabilization of the (nonlinear) Landau-Lifshitz equation are constructed using a Galerkin approximation with 12 linear spline elements. For the following simulations, the parameters are $\nu = 0.02$ and $L = 1$ with initial condition $\mathbf{m}_0(x) = (\sin(2\pi x), \cos(2\pi x), 0)$. Figure 4 illustrates the solution to the uncontrolled Landau-Lifshitz equation settles to $\mathbf{r}_0 = (0, -0.6, 0)$.

Stabilization of the Landau-Lifshitz equation with affine control (8) is illustrated in Figures 5 and ?? with control parameter $k = 0.5$. In Figure 5, the system dynamics are steered from \mathbf{m}_0 to the equilibrium point $\mathbf{r}_1 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$. Figure ?? depicts applying the control twice in succession, forcing the system from the equilibrium \mathbf{r}_0 to $\mathbf{r}_2 = (1, 0, 0)$ and then to $\mathbf{r}_3 = (0, 0, 1)$. In each case, the state of the controlled system converges to the specified point \mathbf{r}_i as predicted by the analysis.

Adding a feedback control so that there is only one equilibrium point also removes hysteresis from the system. Consider the input-output dynamics of the controlled Landau-Lifshitz equation with periodic input $\hat{\mathbf{u}}(t) = (0.001 \cos(\omega t), 0, 0)$. The initial condition is $\mathbf{m}_0(x) = (1, 0, 0)$ and the control parameters are $k = 0.5$ and $\mathbf{r} = (1, 0, 0)$. Figure 7 illustrates the input-output dynamics for $m_1(x, t)$ with x fixed, $L = 1$ and $\nu = 0.02$. It is clear from the figure that persistent looping behaviour does not occur and hence, based on Definition 1, the controlled Landau-Lifshitz equation in (8) does not exhibit hysteresis. Similar behaviour is observed for $m_2(x, t)$ and $m_3(x, t)$.

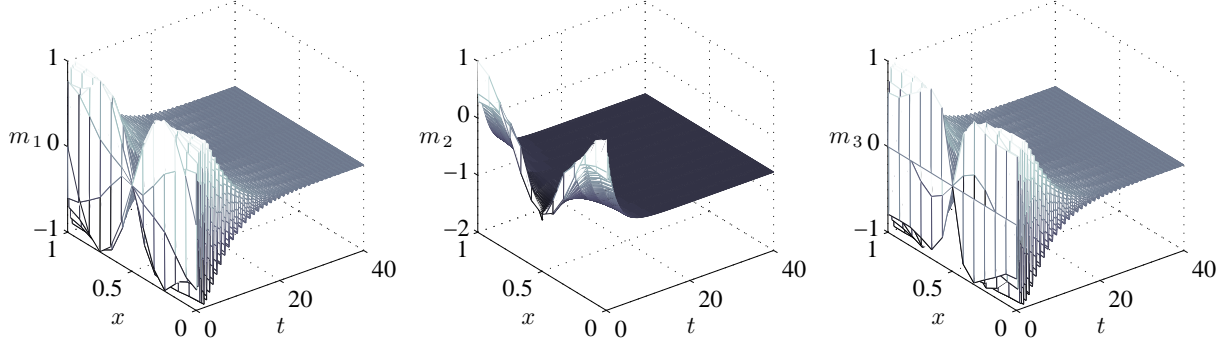


Figure 4: Magnetization in the uncontrolled (nonlinear) Landau-Lifshitz equation moves from initial condition $\mathbf{m}_0(x)$, to the equilibrium $\mathbf{r}_0 = (0, -0.6, 0)$.

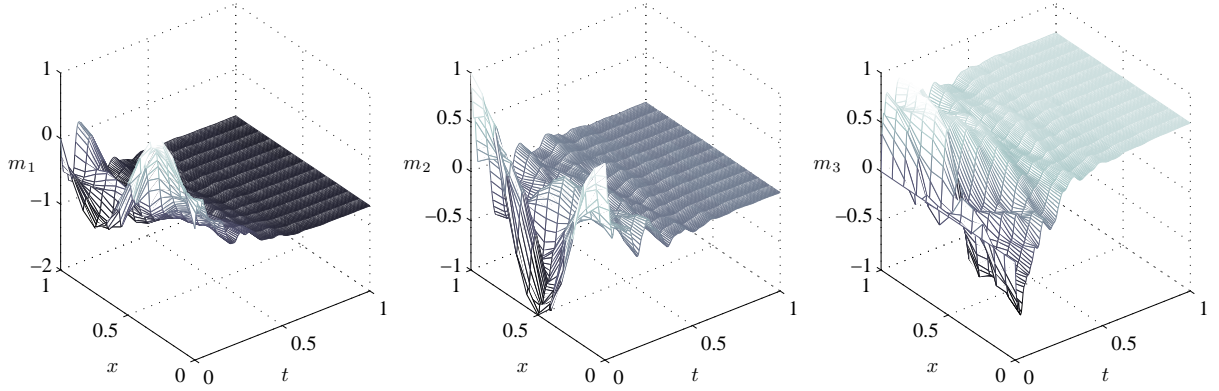


Figure 5: With a proportional control ($k = 0.5$), magnetization in the (nonlinear) Landau-Lifshitz equation with a linear control moves from the initial condition $\mathbf{m}_0(x)$ to the specified equilibrium $\mathbf{r}_1 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

5. Conclusion

The Landau-Lifshitz equation is a nonlinear system of partial differential equations with multiple equilibrium points. The presence of a zero eigenvalue in the linearized equation suggested a simple feedback proportional control can steer the system to an arbitrary equilibrium point. It was then proven that proportional control of the Landau-Lifshitz equation does lead to an equilibrium point that is globally asymptotically stable in the \mathcal{L}_2^3 -norm (Theorem 11) and exponentially stable in the H_1 -norm (Theorem 12).

The fact the Landau-Lifshitz equation is not quasi-linear means linearization is not guaranteed, without further analysis, to predict stability of the nonlinear equation [25]. Moreover, since the objective of the control is to steer between equilibrium points, a linearized analysis, which only yields local results, would not predict stability of the controlled system. Results on preservation of linearized stability require exponential stability of the linearized system; see for example [25, Theorem 3.3] [41, Corollary 2.2][42, Theorem 11.22]. The fact the linearized system is exponentially stable in the \mathcal{L}_2^3 -norm (Theorem 13) is encouraging, but further research is needed to determine

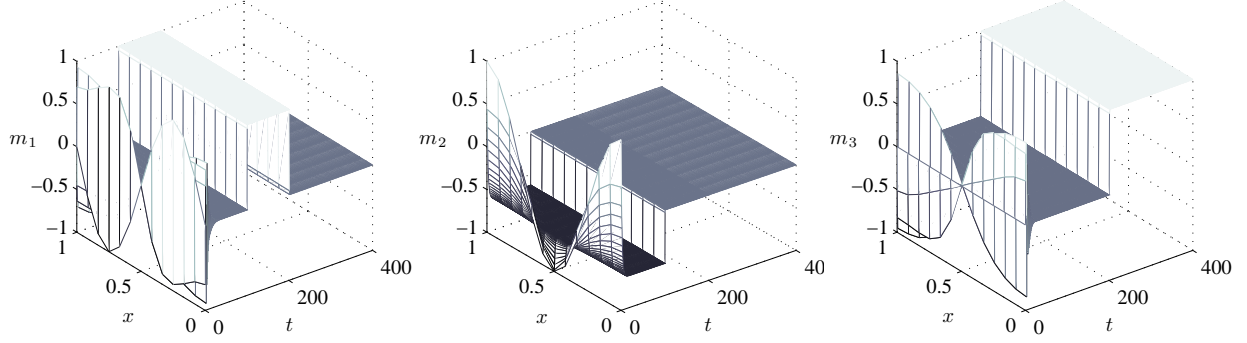


Figure 6: Steering magnetization between specified equilibria with a linear control. The uncontrolled magnetization moves from initial condition \mathbf{m}_0 to $\mathbf{r}_0 = (0, -0.6, 0)$. Proportional control ($k = 0.5$) with two successive values of \mathbf{r} first forces the magnetization to $\mathbf{r}_2 = (1, 0, 0)$ and then to $\mathbf{r}_3 = (0, 0, 1)$.

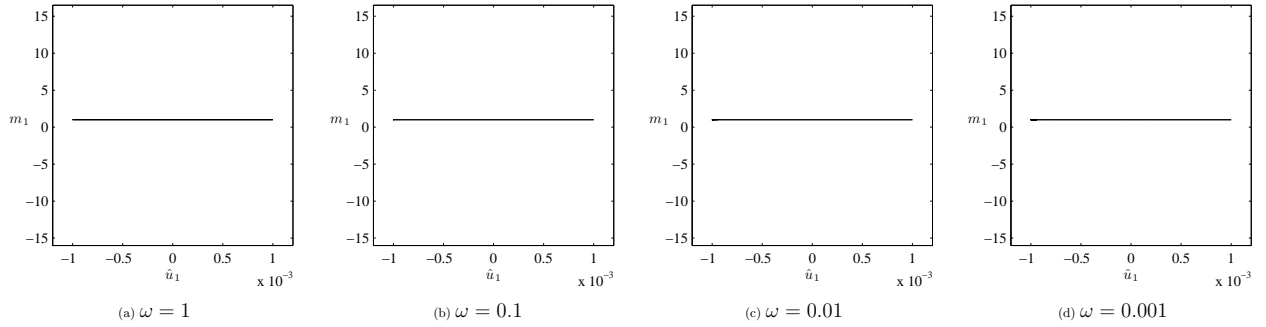


Figure 7: Input–output dynamics for $m_1(x, t)$ of the controlled (nonlinear) Landau-Lifshitz equation in (8) with x fixed demonstrate the absence of persistent looping behaviour as the frequency of the periodic input $(0.001 \cos(\omega t), 0, 0)$ approaches zero. ($L = 1$, $\nu = 0.02$, $\mathbf{m}_0(x) = (1, 0, 0)$, $k = 0.5$, $\mathbf{r} = (1, 0, 0)$)

whether the controlled nonlinear system is also exponentially stable.

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6. References

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